THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
PROF. E. T. WHITTAKER, Sc.D., F.R.S.

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The Mathematical Association.

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INTERNATIONAL CONGRESS OF MATHEMATICIANS. 1928.

THE next International Congress is to be held at Bologna in September of next year. For the first time since the Cambridge Congress of 1912, all differences of nationality are to be disregarded, and mathematicians who believe that in spite of all that has happened since 1912 international cooperation is possible, or even that because of all that has happened international cooperation is imperative, will be glad to have ample warning in order not to make plans that might inadvertently interfere with their attendance.

BRITISH ASSOCIATION.

CONSIDERING the setting for this year's meeting of the British Association, to doubt whether Section A will provide a subsectional programme attractive to readers of the *Gazette* is to overlook the notorious energy of Yorkshire mathematicians. Details are not yet available, but people who enjoy social encounters and holiday mathematics can be sure of a pleasant week at Leeds from August 31st to September 7th.

THE TOTAL ECLIPSE.

A 16-PAGE pamphlet which has been prepared at Burnley Grammar School and published there at the price of 6d. under the title *Guide to the Eclipse of June 29th, 1927*, fits into the gap between the instructions of the professional astronomer and the advice of the professional journalist. Of the five sections—the cause, what to look for, hints to photographers, the eclipse in history, an expedition which "failed"—perhaps it is the third which is the most important; the last section, a quotation from Piazzi Smyth's account of the terrestrial effects of totality when the sun itself is clouded, is a valuable antidote when meteorological friends have operated with their usual depression and we feel that the chance of a fine morning would hardly justify us in getting up early even if we lived on the central line.

The plan which serves as frontispiece is designed to show the relation of Burnley to the region of the shadow, and gives details of the belt of totality only for a few miles southwestward from Settle, but readers of the *Gazette* who are interested in the eclipse will hardly need to be told that the Ordnance Survey Office has published for the occasion a map on which the whole track is marked.

ECLIPSE CALCULATIONS.

AN ELEMENTARY TREATMENT FOR SCHOOLS.

By T. T. Whitehead, M.A.

Mathematics of the Solar Eclipse.

1. A solar eclipse occurs whenever the sun, moon and earth have their centres so nearly in line that the moon hides some part, or all, of the sun from an observer on the earth's surface. In the former case the eclipse is partial, in the latter total. The relative motions of the three bodies are extremely complicated; but as knowledge of the apparent positions of the sun and moon is of importance for safe navigation, these are regularly computed in advance and published by the Admiralty in the *Nautical Almanac*. Since the *Nautical Almanac* tables are the data for eclipse calculations, it is essential to understand the method there employed for recording the positions of the heavenly bodies, and the meanings of a few technical terms.

2. Imagine an observer placed at the centre of the earth, with his head towards the north pole and his feet towards the south pole. Our observer is not supposed to participate in the earth's daily rotation, but is supposed to be carried by the earth in its annual revolution, without rotating, round the sun. If now the observer is endowed with such powerful sight that the earth is perfectly transparent to him, he will appear to occupy the centre of a hollow sphere, on the surface of which he will see the sun, moon and stars. The stars will appear fixed, but the sun and moon will appear to move among the stars, in consequence of the earth's motion round the sun, and the moon's motion round the earth. He will be able to record the position of any object he sees by a system of coordinates very similar to the latitude and longitude used by the geographer to fix the position of a place on the earth's surface.

Thus the point directly overhead is the celestial north pole, the point directly underfoot the celestial south pole, and the great circle half-way between them the celestial equator. The angular distance of an object north or south of the celestial equator is known not as latitude, however, but as declination, and the coordinate corresponding to longitude is called right ascension. Furthermore, the right ascension is usually given in hours, minutes and seconds, as this is the form in which it is most useful to the navigator; like longitude it may be converted into degrees, minutes and seconds at the rate of 15 degrees per hour of right ascension. Owing to the motions of the earth and moon in their orbits, the apparent declinations and right ascensions of the sun and moon change constantly. The *Nautical Almanac* gives their values for each noon in the case of the sun, but hourly in the case of the moon, for which they alter much more rapidly.

3. Since the observer at the centre of the earth has no means of judging the radius of the sphere which apparently surrounds him, it is most convenient to record the sizes of the sun and moon in terms of the angles their discs subtend at his eye. The *Nautical Almanac* therefore gives the angular "semidiameters" for each noon in the case of the sun, and for each noon and midnight in the case of the moon. In the case of the sun, its semidiameter as seen from the centre of the earth is accurate enough (for our purpose) for any point on the earth's surface; but the moon's semidiameter may be appreciably changed by moving nearer to the moon through the distance of the earth's radius. The amount of this change, for any point on the earth's surface, depends upon the height of the moon above the horizon, and is tabulated in books on navigation, and also in many editions of mathematical tables, under the title of "Augmentation of Moon's Semidiameter." (See Table C).

4. The moon and sun are much nearer to our earth than the stars. Consequently if our observer moves from the centre of the earth to its surface, he will generally change the apparent positions of these two bodies on the surface of the sphere.

The greatest change in apparent position is made when the observer moves from E, the earth's centre, to B, a point on the equator such that EBM is a right angle. The change is somewhat less than this if the observer moves towards the pole, because the polar radius of the earth is less than the equatorial

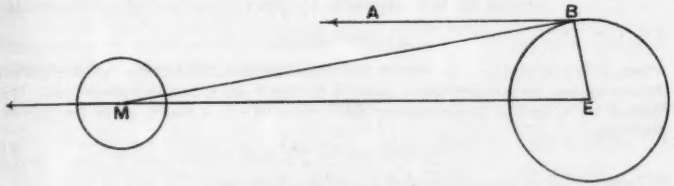

FIG. 1.

radius. In the diagram the angle ABM measures the change of apparent position, and is called the horizontal equatorial parallax. Since the angle ABM is equal to BME, the horizontal parallax gives the angular semidiameter of the earth as seen from M.

The parallax clearly depends upon the distance EM, which is constantly changing; so the *Nautical Almanac* tabulates its value, for every tenth day in the case of the sun, and twice a day in the case of the moon.

The horizontal parallax being thus known for the equator, its value for any other latitude is found by multiplying by the earth's radius at that latitude, the equatorial radius being taken as unity.

5. We have now prepared the way for the consideration of the eclipse problem proper. Here two questions arise: in the event of a solar eclipse, over what parts of the earth will it be visible, and, in the case of any such point, at what times will the partial and total eclipses begin and end? The most convenient way of answering these questions is from the consideration of a figure, supplied partly by our observer at the earth's centre, and partly by a colleague whose station is at the centre of the sun. Let us suppose that each of these observers is supplied with a camera, and that each points his lens directly along the line joining the earth's centre to that of the sun. Let us further imagine that the observer at the sun's centre employs a telescopic lens, so that the moon in his photograph is of the same size as in the photograph of his earthly colleague. Then in the centre of our earthly photographer's picture is the centre of the sun, but the centre of the solar photographer's plate is occupied by the centre of the earth. The moons on the two plates can be made to coincide by turning either of the plates over. Then, if the photographs were both taken at the same instant, the position where the centre of the moon's shadow falls on the earth at that instant will be clearly shown by combining the two photographs.

6. The photography is clearly impossible, but it is possible to construct mathematically the essential parts of the picture, and furthermore to fix the positions of important points on the diagram in terms of the usual cartesian coordinates. For this purpose we take the common centre of the earth and sun as the origin, O, the Y axis along the line joining O to the position of the north pole of the earth, and the X axis at right angles as usual. If we reverse the photograph of the earthly observer, the positive X axis is on the usual side of the Y axis. The contribution of the observer at the centre of the earth is

the position of the moon on our diagram. Let us make the radius of the moon equal to the number of seconds given in the almanac as its semidiameter; then, if

A = moon's R.A. in seconds of arc D = moon's declination in seconds

a = sun's R.A. „ „ d = sun's „ „ „

we get, for the coordinates of the moon's centre, since circles of R.A. converge towards the poles exactly like circles of longitude,

$$X = (A - a)\cos D, \quad \text{.....6·1}$$

$$Y = (D - d) + C, \quad \text{.....6·2}$$

where C is a correction necessary owing to the fact that circles of declination do not appear as straight lines parallel to the X axis, but as hyperbolas. It is proved by spherical trigonometry that when $A - a$ is small, as in the present problem,

$$C = 206265 \sin 2D \sin^2 \tfrac{1}{2}(A - a). \quad \text{.....6·3}$$

Table B below was calculated from this formula.

7. The coordinates x, y, of the representation of a point in lat. l, long. L, on the earth's surface are contributed by the observer on the sun. Formulae for their calculation may be deduced from a consideration of the diagrams given below.

Fig. 2A represents the earth as seen from the sun. The pole of the earth will fall on the line OY, but not at V as a rule. To represent OW correctly to scale, we must remember that it will not appear so large as if the moon and earth were the same distance from the sun, and hence the number of seconds in the moon's equatorial parallax must be diminished. A little consideration leads to the conclusion that the figure of the earth is an ellipse, with centre O, and constructed to scale, so that its greatest radius vector is equal to the number of seconds in the difference of the equatorial parallaxes of the moon and sun. In consequence of the earth's rotation, any point Q on the surface of the earth will appear to trace out an ellipse such as BAB_1A_1 every 24 hours.

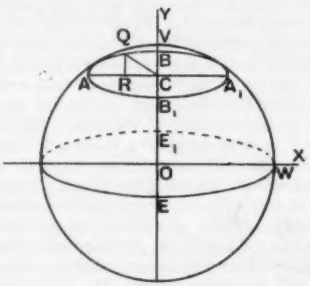

FIG. 2A.

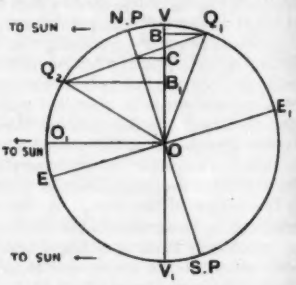

FIG. 2B.

We can find the lengths of the major semiaxis AC and the minor semiaxis CB, and the position of the centre C by the help of Fig. 2B. In this figure we have a section sideways through the earth, so that the line VOV_1 divides the sunlit from the dark hemisphere. E and E_1 are points on the equator, and consequently, by § 4,

$EO = OE_1$ = (moon's equatorial parallax − sun's equatorial parallax)

$= P$, say,

$$\left.\begin{aligned} EOO_1 &= d \text{ (sun's declination)},\\ EOQ_2 &= l = E_1OQ_1 \text{ (geocentric latitude of } Q),\\ OQ_2 &= OQ_1 = P \times \text{(earth's radius at latitude } l) = p, \text{ say.} \end{aligned}\right\} \quad \ldots\ldots 7{\cdot}1$$

Consequently

$$OB = p \sin (l + d), \quad \ldots\ldots 7{\cdot}2$$

$$OB_1 = p \sin (l - d), \quad \ldots\ldots 7{\cdot}3$$

and since $BC = B_1C$, we get for the minor semiaxis of the ellipse

$$BC = p \cos l \sin d, \quad \ldots\ldots 7{\cdot}4$$

and since also $OC = \frac{1}{2}(OB + OB_1)$, we get

$$OC = p \sin l \cos d. \quad \ldots\ldots 7{\cdot}5$$

Now Q_1 is the position of Q at midnight and Q moves uniformly along the circumference of the circle of latitude of radius $p \cos l$.

This circle is the auxiliary circle of the ellipse BAB_1A_1, and therefore the coordinates of Q relative to C are, by the properties of the auxiliary circle

$$\xi = -AC \sin t, \quad \ldots\ldots 7{\cdot}6$$

$$\eta = \quad BC \cos t, \quad \ldots\ldots 7{\cdot}7$$

where t is the time from midnight, reduced to degrees in the usual way. Since $y = OC + \eta$, and $x = \xi$, we get for the coordinates relative to O,

$$x = -p \cos l \sin t, \quad \ldots\ldots 7{\cdot}8$$

$$y = p \sin l \cos d + p \cos l \sin d \cos t. \quad \ldots\ldots 7{\cdot}9$$

8. The latitude l used in § 7 is the geocentric latitude, and differs from the usual or geographical latitude. The geographic latitude is the angle of elevation of the pole of the heavens above the horizon, *i.e.* above the tangent plane which touches the earth at the point.

In figure 3 the flattening of the poles is exaggerated so as to render the meaning more clear. The angle HPN is the geographic latitude, and is plainly equal to the angle PQE, where $HPQ = 90°$.

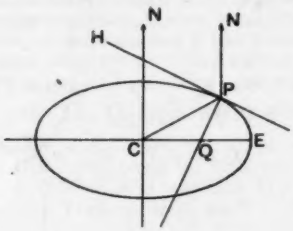

FIG. 3.

Now the angle CPE is the angle required for l in the preceding section, and is less than PQE by CPQ.

The angle CPQ is called the angle of the vertical, and may be calculated for all latitudes from a knowledge of the figure of the earth. It is frequently required in astronomical calculations, and is tabulated in the books of mathematical tables.

The angle t in § 7 is the local apparent time at Q, reckoned from midnight; if Greenwich Mean Time is known, it may be calculated from the formula

$$t = \text{G.M.T.} \pm \text{Equation of Time} - \text{West longitude of } Q, \quad \ldots\ldots\ldots 8{\cdot}1$$

with the usual relationship of 1 hr. of time = 15° of arc.

Conversely, if t is known, G.M.T. is easily found.

9. To calculate the position on the earth where the eclipse is central at any given G.M.T., calculate X, Y as in § 6, and equate to the expressions for x, y, in § 7. We thus get

$$-p \cos l \sin t = X, \quad \ldots\ldots\ldots 9{\cdot}1$$

$$p \sin l \cos d + p \cos l \sin d \cos t = Y. \quad \ldots\ldots\ldots 9{\cdot}2$$

The unknowns are l and t (l is involved in p, but as p never differs much from P, the difference in seconds between the equatorial parallaxes of the moon and sun, approximate values of l are given by putting P for p, after which p can be found and the calculation repeated).

Equation 9·1 is equivalent to

$$\sin t = -\frac{X}{p \cos l}, \quad \ldots\ldots\ldots 9{\cdot}3$$

and 9·2 to

$$\cos t = \frac{Y - p \sin l \cos d}{p \cos l \sin d}. \quad \ldots\ldots\ldots 9{\cdot}4$$

Since $\sin^2 t + \cos^2 t = 1$, t can be eliminated by squaring both and adding. This leads after simplification to the quadratic equation in $\sin l$,

$$p^2 \sin^2 l - 2pY \cos d \sin l + (X^2 - p^2) \sin^2 d + Y^2 = 0,$$

the solutions of which are

$$\sin l = \frac{Y \cos d \pm \sin d \sqrt{p^2 - X^2 - Y^2}}{p}. \quad \ldots\ldots\ldots 9{\cdot}5$$

This gives two values of l, only one of which will give t for the sunlit hemisphere and satisfy both 9·3 and 9·4. When t has been determined, the longitude of Q follows, since the G.M.T. is known.

10. A belt of width

$$w = (\text{moon's apparent semidiameter} - \text{sun's semidiameter})$$

on each side of the central line is in total eclipse.

To find the limits of totality when the eclipse is central at Q, let dX and dY be the rates of change of X and Y per unit time.

(A convenient unit is 1 minute.)

Let $dS = \sqrt{dX^2 + dY^2}$, then, by similar triangles in Fig. 4,

$$Q_1R : QR : QQ_1 = dX : dY : dS;$$

$$\therefore\ Q_1R = w\frac{dX}{dS},\quad QR = w\frac{dY}{dS}.$$

So that if $Y + w\dfrac{dX}{dS}$ is put for Y

and $X - w\dfrac{dY}{dS}$ is put for X $\quad \ldots\ldots\ldots 10{\cdot}1$

in 9·3, 9·4 and 9·5, the latitude and longitude of Q_1 can be determined. The corresponding point on the southern boundary of totality is found by putting, in the same formulae,

$$Y - w\frac{dX}{ds} \text{ for } Y \quad \text{and} \quad X + w\frac{dY}{ds} \text{ for } X. \quad \ldots\ldots\ldots 10{\cdot}2$$

The limits of partial eclipse are similarly found, w being replaced by W, the sum (in seconds) of the apparent semidiameters of the sun and moon.

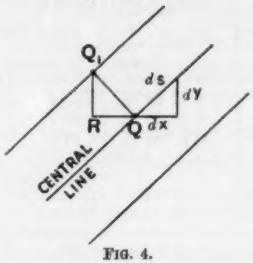

FIG. 4.

Since the points are required to be on the earth's surface, the semidiameter of the moon used in this section should be "augmented for altitude *".

11. To calculate the times of the different phases of the eclipse, and the angles of contact from the north, for a particular place Q, it is only necessary to notice that each observer on the earth sees (on the figure 2A) the moon as a circle with centre X, Y, and radius equal to the augmented semidiameter, and the sun as a circle with centre on the line joining his own position to the centre of the sun, *i.e.* at x, y. The radius of the sun is the sun's semidiameter as given in the almanac.

For a G.M.T. near to the appropriate time given by §§ 9, 10, calculate X, Y, dX, dY, x, y, and also dx, dy from the equations (found by differentiating 7·8 and 7·9),

$$dx = -p \cos l \cos t \, dt, \quad \ldots\ldots 11{\cdot}1$$

$$dy = -p \cos l \sin d \sin t \, dt, \quad \ldots\ldots 11{\cdot}2$$

where dt may be conveniently taken to be 1 minute of time reduced to circular measure ($=\cdot 004360$ radian).

Then the coordinates of the point of central eclipse at the assumed time, relative to the coordinates of Q, are $(X-x,\ Y-y)$ and 1 minute later are $(X-x+dX-dx,\ Y-y+dY-dy)$.

The line joining these points is plotted on a separate diagram (or its equation found), and the coordinates (h_1, k_1), (h_2, k_2) are found, for the points P, P' of its intersection with a circle, whose centre is at the origin, and whose radius is the difference of the apparent semidiameters of the sun and moon. Then the time of contact is

$$\frac{h(X-x)}{dX-dx} \text{ minutes after epoch,} \quad \ldots\ldots 11{\cdot}3$$

where $h=h_1$ for the earlier, and $h=h_2$ for the later contact in the case of the total phase. (For the partial phase we must substitute the sum of the apparent semidiameters). The angle of contact is YOP,

$$=\tan^{-1}\frac{k}{h}, \quad \ldots\ldots 11{\cdot}4$$

from the north point, remembering that for the total phase the first contact is the more easterly, but for the partial phase, the more westerly of the pair.

* The altitude of the moon may be assumed, for this purpose, to be equal to that of the sun, and is given by the formula

$$\sin a = \sin\lambda \sin d - \cos\lambda \cos d \cos t,$$

when a is the altitude, and λ is the geographical latitude of the station: d and t have the same meanings as in § 9.

Tables for the Eclipse of 29th June, 1927.

(A) R.A. and Declination of Sun and Moon.

G.M.T.	Sun	
	R.A.	Dec.
June 29. 5 h. 24 m.	6 h. 28 m. 13·83 s.	23° 17′ 24·7″ N.
5 h. 25 m.	6 h. 28 m. 14·00 s.	23° 17′ 24·6″
5 h. 26 m.	6 h. 28 m. 14·17 s.	23° 17′ 24·5″

G.M.T.	Moon	
	R.A.	Dec.
June 29. 5 h. 24 m.	6 h. 25 m. 46·97 s.	24° 02′ 55·5″ N.
5 h. 25 m.	6 h. 25 m. 49·46 s.	24° 02′ 57·0″
5 h. 26 m.	6 h. 25 m. 51·95 s.	24° 02′ 58·6″

At 5 h. 25 m. Equation of Time = 3 m. 01·68 s. (Add to apparent time to get mean time). Sun's semidiameter = 15′ 43·9″.

(B) Values of C (see § 6·2).

($D = 24° 02′$).

$A-a$	C	$A-a$	C
2 m. 00 s. *i.e.* 30′	2·9″	2 m. 40 s. *i.e.* 40′	5·2″
2 m. 20 s. *i.e.* 35′	4·0″	3 m. 00 s. *i.e.* 45′	6·6″

(C) Augmentation of Moon's Semidiameter with Altitude.

(Semidiameter = 15′ 47·70″) at 5 h. 25 m.

Altitude	Augmentation	Altitude	Augmentation
6°	1·80″	14°	3·98″
8°	2·35″	16°	4·52″
10°	2·90″	18°	5·05″
12°	3·44″	20°	5·58″

(D) Logarithm of Earth's Radius and Angle of the Vertical.

Lat.	Log R	Angle of Vertical	
52° 00′	$\bar{1}$·999094	−11′ 16·6″	E.
52° 30′	$\bar{1}$·999081	−11′ 13·0″	Sun's equatorial parallax
53° 00′	$\bar{1}$·999069	−11′ 09·5″	= 8·66″
53° 30′	$\bar{1}$·999057	−11′ 05·9″	
54° 00′	$\bar{1}$·999045	−11′ 02·3″	
54° 30′	$\bar{1}$·999033	−10′ 58·7″	Moon's equatorial parallax
55° 00′	$\bar{1}$·999020	−10′ 55·1″	= 57′ 58·42″
55° 30′	$\bar{1}$·999008	−10′ 51·5″	

T. T. Whitehead.

CAN THE RANGE OF GEOMETRY TAUGHT IN SCHOOLS BE WIDENED?

By Professor H. F. Baker, F.R.S.

The following, written at the request of the Editor of the *Gazette*, gives an account of the mathematical part of the remarks made by the lecturer at the Annual Meeting of the Association last January. There are added some words dealing with geometry on a sphere, in regard to which some questions were raised at the meeting.

Schoolchildren are taught that Geometry is a collection of properties for the mensuration of figures. No objection to this as a preliminary was suggested, provided the major part of the child's energy is not devoted to learning to reproduce the formal proofs of the equalities. When this is so, experience seems to show that this may lead, in the words of one speaker, to the children *loathing* the subject called Geometry.

But the question was raised whether the children could not also be taught parts of descriptive geometry; this would arise naturally as a development of a course of geometrical drawing and model making, which in any case seems a proper preliminary to systematic geometry; moreover it is only fair to the child that he should be given the opportunity of seeing that metrical properties are particular cases of descriptive properties, as is undoubtedly the case. At present the exact opposite is often taught.

It would seem that the only two things, from the child's point of view, in which such a training in the elements of descriptive geometry need differ from what he is now taught are: (1) the assumptions that any two lines in a plane meet in a point, that any line meets any plane, and that any two planes meet in a line—probably these seem to a child more natural than the assumptions as to parallel lines; (2) the free use of three dimensions, of projection in three dimensions, and of section by a plane. It seems plausible that the properties of three dimensions are prior, in a child's mind, to the properties of plane figures; if this be so, the plan suggested is proper. But if not, one of the main practical reasons for teaching geometry is to train a facility in handling figures in space, so that the study of models should be an early part of the subject.

By actual experience with figures and models the child can see that many propositions, when projection is allowed, follow at once from the fundamental intersections of lines and planes (the classical example is Desargues' proposition in regard to two triangles in perspective)—follow, as we say for brevity, from the *Propositions of Incidence*. But there are others which do not so follow; which require a certain further proposition. It is a discovery of great interest that, beside the propositions of incidence, there is only one single property which need be assumed. Of the truth of this property the child can convince himself experimentally; but he may be told that it is impossible to deduce

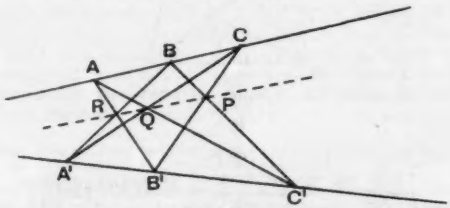

Fig. 1.

it from the propositions of incidence, though it can be presented in various forms in virtue of these propositions. This property is probably presented

in the most convenient form by what is called *Pappus' Theorem*. This is, that if A, B, C be three points on a line, and A', B', C' be three points on another line lying in the same plane with the former, and if P, Q, R be the respective intersections $(BC', B'C)$, $(CA', C'A)$, $(AB', A'B)$, then P, Q, R are in line.

It is not suggested that the child should be taught any logical proof of Pappus' theorem, but only that he should be taught to recognise its importance. Those teachers, however, who regard it as necessary to put the theorem into connexion with metrical geometry can prove it by reducing it to showing,

FIG. 2.

in the diagram annexed, that CP, which is not assumed to pass through A, passes through the intersection of DE and BF; which follows (!) (*a*) from the parallelogram of forces or (*b*) by proportional parallel triangles. It is needless to repeat that in the lecturer's opinion either "proof" is absurdly illogical.

In the lecture three definite proposals were put forward: (1) that children should be shown that propositions relating to middle points are particular cases of propositions relating to harmonic points, (2) that they should be shown also that propositions relating to perpendicular lines are particular cases of propositions relating to lines which are harmonic in regard to two given points, (3) that they should be practised in determining whether propositions depend on Pappus' theorem or not.

In illustration of what is meant, consider in turn the two elementary theorems (1) that the medians of a triangle meet in a point, (2) that the lines drawn through the middle points of the sides of a triangle at right angles to these sides meet in a point.

Apparently the proof of the former proposition, which is most elementary from the point of view of Euclidean geometry, is the following: Let the medians BE, CF meet in O; produce OF to G making FG equal to OF; prove

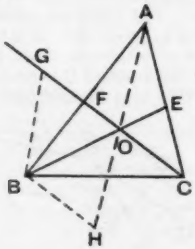

FIG. 3.

that BG is equal and parallel to OA. Then draw BH parallel to FC meeting AO in H. Whence OH, equal to BG, is equal to OA. By a similar proof, if we draw a line through C parallel to EB, it will meet AO in the same point H. Then $BOCH$ is a parallelogram, of which the diagonal OH bisects BC, so that AO is the median from A.

This argument requires the theory of congruent triangles, and hence the axiom of congruence, beside the axiom of parallels, both of which, from the point of view of descriptive geometry, seem very arbitrary.

The proof of the proposition of descriptive geometry of which the foregoing is a particular case is everyway as brief as the proof given, and is very much simpler in that it requires nothing arbitrary, following only from the obvious propositions of incidence. It requires a certain familiarity with the properties of the point which is the harmonic conjugate in regard to two given points A, B of a line, of a third point R, of this line; but this familiarity is possible at the very outset of the subject. For brevity, let the harmonic conjugate of R, in regard to A and B, be denoted by $(A, B)/R$. The general proposition which is in question is this: let an arbitrary line meet the sides BC, CA, AB, of a triangle, respectively in P, Q, R. Take on these sides, respectively, the harmonic conjugates denoted, as above, by $(B, C)/P$, $(C, A)/Q$, and $(A, B)/R$. Let these be D, E, F. It is required to prove that AD, BE, CF meet in a point. We follow the lettering of the particular case proved above.

Let BE, CF meet in O. Let AO meet the line PQR in K, and let KB meet CF in G. Again let FC meet PQR in M, and let BM meet AO in H. We wish to show that if BE meets PQR in L, then LC passes through H, and to infer that AO contains the point D, or $(B, C)/P$.

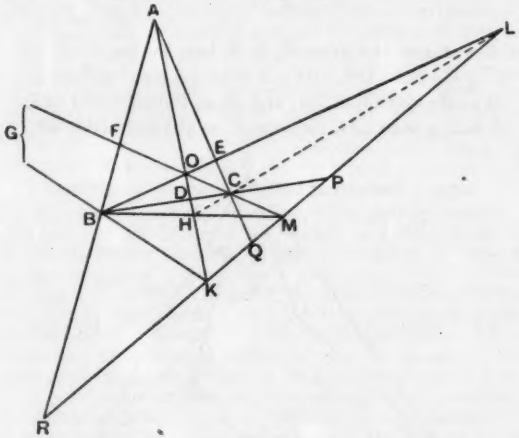

FIG. 4.

Now $F=(A, B)/R$ is the same as $B=(F, R)/A$, which, by projection from M, gives $H=(O, K)/A$. Thence, by projection from L, the line LH meets AQ in $(E, Q)/A$, which, however, is C. Thus L, C, H are in line. Then, as OC, BH meet in M, and BO, HC meet in L, it follows from the definition of the harmonic conjugate, that OH, which is AO, meets BC in $(B, C)/P$. And this completes the demonstration.*

*If we were not illustrating the generalization of the ordinary metrical proof, we could say simply: (1) As $AFBR$ and $AECQ$ are harmonic ranges, the points P, E, F are in line; (2) Thence, by definition, AO contains the harmonic conjugate, D, of P, in regard to B and C. And this remark shews up nicely the cumbrousness of the metrical theory of mid points. For the two results; (1) that the join of the mid points of two sides of a triangle is parallel to the third side; (2) that if in a quadrilateral (trapezium) $BCEF$, the opposite sides BC, FE be parallel, while BF, CE meet in A, and BE, CF meet in O, *then* AO bisects BC and FE, do not lie at the beginning of the subject. To a boy who has appreciated the point of view here propounded these results would be obvious.

Consider now the second proposition referred to, that the lines bisecting the sides of a triangle at right angles meet in a point. The generalised proposition is as follows: Let A, B, C be any triangle; let I, J be any two points; let the line IJ meet the sides BC, CA, AB respectively in P, Q, R; let, as before, D, E, F be the points $(B, C)/P$, $(C, A)/Q$ and $(A, B)/R$, respectively. We are to draw through D the line to the point $(I, J)/P$, to draw through E the line to the point $(I, J)/Q$, and through F the line to the point $(I, J)/R$. Then we are to show that these three lines meet in a point.

The construction of D and the line drawn through it can be made together, by taking L, the intersection of BI, CJ and L', the intersection of BJ and CI. Then, by the definition of the harmonic conjugate, the line LL' goes through D and through $(I, J)/P$, and is one of the three lines which are to be shown to meet in a point.

FIG. 5.

Thus, we do not need the points P, Q, R, but defining

L as the point (BI, CJ) and L' as the point (BJ, CI),

M as the point (CI, AJ) and M' as the point (CJ, AI),

N as the point (AI, BJ) and N' as the point (AJ, BI),

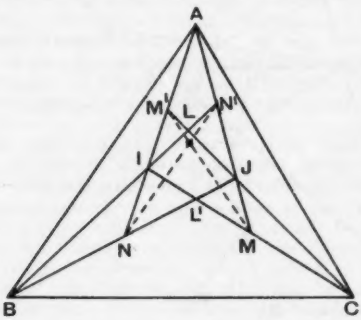

FIG. 6.

we are to show that LL', MM', NN' meet in a point. This however is Pappus' theorem for the two triads of collinear points NIM' and MJN'.

The proposition of the intersecting perpendiculars is thus, from the present point of view, entirely different from the proposition in regard to the intersecting medians.

It may be remarked however that both propositions may be regarded as particular cases of another, of which the proof depends in the general case on Pappus' theorem. For it is easy to see from the definition of the harmonic conjugate that in both cases the lines EF, FD, DE pass respectively through P, Q, R. Now it can be shown that if the sides of a triangle DEF meet a line respectively in P, Q, R, and lines be drawn, DP_1, EQ_1, FR_1, to meet PQR

in points P_1, Q_1, R_1 respectively, such that P, P_1 and Q, Q_1 and R, R_1 are, all three, pairs of harmonic conjugates in regard to the same pair of points I, J of the line PQR, then DP_1, EQ_1, FR_1 meet in a point. This, like Pappus' theorem, is a theorem of Greek geometry; it can be shown that it does not follow from the propositions of incidence, but requires Pappus' theorem.

The proposition of the medians, and the second proposition of the bisecting perpendiculars, are known to hold for a spherical triangle. A remark may be added as to their proof from the present point of view. In the first place, the theory of the fourth harmonic conjugate formed from three points of a great circle, and Pappus' theorem, both hold on the sphere, as is obvious by projection from the centre on to a plane. We may then define the mid-point of an arc BC as the point D such that if P, $=(B, C)/D$, be its harmonic conjugate, then DP is a quadrant. (There are of course two points P, but they are both on the same diameter of the sphere.) It follows then that if D, E, F be the mid-points of the sides BC, CA, AB, and BC meets EF in P, P', while CA meets FD in Q, Q' and AB meets DE in R, R', then $PP'QQ'RR'$ are on a great circle; this arises by projection from the corresponding proposition in a plane. The proposition of the medians has essentially the same proof as for a plane. The bisecting perpendiculars clearly all contain a point whose distances from P, Q, R are each a quadrant; namely, they meet in the pole of the great circle $PP'QQ'RR'$. They meet this great circle in points P_1, Q_1, R_1 such that PP_1, QQ_1, RR_1 are quadrants. Thus there are two points I, J (which are imaginary) upon this great circle, in regard to which each of PP_1, QQ_1, RR_1 consists of harmonic conjugates. The generalised proposition, in which we take two quite general, real, points on this great circle, for I and J, and construct P_1, Q_1, R_1 accordingly, as in the plane case, is proved precisely as in that case. And the general result stated for a plane, including this proposition and the proposition for the medians, holds equally here.

H. F. BAKER.

DISCUSSION FOLLOWING PROF. BAKER'S PAPER.

Mr. **W. C. Fletcher** opened the discussion by urging practical teachers who would appreciate the difficulties of the subject to express their views. He then asked, Did the Pappus theorem depend on parallelism? The proof which Professor Baker happened to give made it so depend.

Professor **Baker** said that the proof for school purposes of the Pappus theorem was that two projective ranges had two corresponding points. The result depended upon the solution of a quadratic equation.

Mr. **Fletcher** said that he could not quite grasp the answer. It was simply a good illustration of the enormous chasm that lay between most of them there—certainly himself—and the people who really knew geometry.

Professor **Baker** said that he had definitely put it like that in order to emphasize what he had said. He did not think the Pappus theorem could be proved for the sake of the boys in school; that, he thought, would be going too far.

Mr. **Fletcher** said that the point that had interested him was this. He had run his head very hard indeed against the question of medians, and he came across this paradox. Every proof he had ever heard of for the intersection of medians did depend on parallels. That was the source of interest to him. On more general grounds he thought that all that any one of them could follow in what Professor Baker had said must have impressed them with its extreme beauty, its attractiveness, something not to be found in geometry as most of them conceived it, but just the thing that they wanted to find. If they could get somewhere nearer to that sort of work—work that all of them had dabbled in at some time—it would be all to the good, but it was a long, long job to get the ordinary schoolboy up to that. He was relieved that Professor Baker agreed that they would have to take the boy through

elementary algebraic geometry first. Certainly that had first to be done. The difficulty was that they could not get more than one boy in ten to have any real power of thought on the subject. If it was possible only for that happy residuum which did make some mastery of Euclid to go on to this matter there would be hope of getting it done. It was possibly good sixth form work, but as to third form work he had his doubts.

Professor **Lodge** said that the little duel between the two principal antagonists had been amusing. Most of those present, he thought, were likely to agree with Mr. Fletcher that it would be the super-schoolboy rather than the average schoolboy for whom these extensions would be useful. In most cases this descriptive geometry was postponed—at all events it was so in his time—to the university period. It was not in his time taught in schools, but in many schools now a certain amount of descriptive geometry was taught, and it would be rather interesting to get the experience of some of those who did teach the subject there.

A **Member** said that the main reason why the extensions should be taught in the school was that unless they were taught at school they were not taught at all. They were not taught in the majority of the universities, not even in Professor Baker's university (Cambridge). Therefore it was essential to try and do something in the schools which was not done at the universities. There were, of course, disadvantages. Certain pupils of his had gone to universities and had used delightfully simple proofs, depending, for instance, upon the only natural definition of a harmonic range, and had been severely reprimanded for not producing a long and complicated proof embodying some ratio being equal to -1.

Another **Member** asked Professor Baker further to elucidate the remark he had understood him to make that geometrical theorems depended either on the graphical method or on the theorem of Pappus.

Professor **Baker** said that it was quite literally true that every geometrical theorem was either deduced from the propositions of incidence, for which he had no better name than graphical theorems, or from the theorem of Pappus, which, in the graphical geometry, was representative of the notions of continuity, or, if they pleased, of the fact that a line was a metrical thing, or again, as he had put it to Mr. Fletcher, of the fact that for two projective ranges there were two common corresponding points. It was literally the case that there were only those two bases of their geometrical theorems. A good deal of development had already proceeded on those lines.

The **Member** asked whether if great circles on the surface of a sphere were substituted for straight lines, the Pappus theorem remained true.

Professor **Baker**: If you project the Pappus figure on a plane from the point of a sphere—

The **Member**: What I really meant was whether if you take two great circles, mark three points on one and three points on the other, join those points by great circles as you join them by straight lines, the theorem holds?

Professor **Baker**: Yes, I understand; I do not think it does.*

Professor **A. R. Forsyth** said that he was in much the same difficulty as that from which Professor Baker said at the beginning of his remarks he was suffering. He had never been a schoolmaster or attempted to teach schoolboys, and he thought he had not even examined them. But he had had occasion in his earlier days, on the staff of his college, to teach those who had been schoolboys, or so he believed, though he could not gather from what they told him that they had learned anything as schoolboys; and he had to make a beginning in his own fashion. The difficulty he found with these students, who some day aspired to go in for what was called "the General" examination, was that he seemed to have nothing to build upon; and, there

* See the concluding remarks of the foregoing note.

as elsewhere, personally he derived enormous benefit from the experience, because he learned how to teach. But the difficulty was to get into the mind and thought, so far as there was active thought, of the scholars, and reach something upon which to build, either, it might be, Euclid as they called it, or a little algebra. But there did not seem to be anything upon which to build. They had certainly got hold of no principles, and they had had uncommonly little practice. Therefore, when teaching this—if he might call it so—residuum of the mathematicians, he would not have ventured to suggest to them that there ever was such a person as Pappus! They, of course, like so many others in their day and since, were forced to get through a certain amount—a minimum that would be satisfactory—in order to pass some "loathsome" examination. All that they wanted was the minimum necessary for the purpose. As for having an interest in the subject, especially the rudiments, when it was a compulsory subject in an examination that they loathed, this was never thought of. The difficulty was to get at them. If one could once have overcome that particular difficulty, one might have been able to do something by way of teaching them even the very rudiments of principles.

The speaker had an earlier experience in his short period at what was then University College and was now the University of Liverpool, when he had to train students of a somewhat better average calibre. They had had to pass an adequate entrance examination, and were going on for what purported to be a real mathematical examination. But again there was the curse of the examination over it. They wanted to learn just enough to enable them to answer questions. As for an understanding and grasp of the subject, that was not their ambition at all. What they wanted was to get through the examination and be done with the whole thing. Unless and until teachers could do something to interest their pupils—boys and girls, and, he would imagine, girls more easily than boys—in the substance of the thing for its own sake, and not for the sake of an examination, he was afraid their work was a little like throwing seed on stony ground. The difficulty was to try and interest the students, boys and girls, in the foundations and beginnings of things, and interest them in such a way that they could see, as they devoted themselves to the subject, that something was coming out of it—nothing connected with examinations, but something connected with new knowledge, something that they had not suspected, something perhaps a little surprising, something new, not something that they had to "get up." If there was the slightest suggestion on their part that a little active effort made by them led to some created thing, that there was something of their own handiwork to show for it, then they might have a sense of having themselves achieved something, instead of having had things stuffed into them, which latter was the feeling that most of them had.

So much for a beginning: but he was afraid even his slightly interested pupil would have to wait a very long time before he could get to the stage of having taught to him a proposition so fundamental, so thorough, and so far-reaching, as the one that Professor Baker had enunciated, that just as Gaul was divided into three parts, so all geometry was divided into two. It was, no doubt, a magnificent conception—certainly for professors. Professor Baker, however, must momentarily regard him as "the rather intelligent man in the omnibus." He (the speaker) did not think that, save for the very exceptional Macaulay schoolboy, it would ever be possible to secure in schools anything like the grasp of a principle so broad and expansive and generalized as Professor Baker had brought forward. Wise it was, perhaps the quintessence of geometrical wisdom, but, like so much wisdom, perhaps too strong meat for the youthful mind.

Professor Forsyth went on to say that, if he might deal with some of the geometrical work to which Professor Baker referred in detail, he could speak very much in the way Mr. Fletcher had done, only with more painful

memories, because when he was a small schoolboy he loathed this subject with a great loathing. He could look back to the time when they had three geometry lessons a week, and he was punished pretty regularly once a week! He was not met by the presentation of a large principle, but by a much more material object used for the purpose. Any attempt to introduce to his mind a notion of general principles at that stage—and he did not suppose he was much more stupid than the average pupil—would have been effort entirely and completely wasted on the part of the schoolmaster. The schoolmaster did not interest him in the subject, and even if the schoolmaster had had principles that would have solved the universe, it would have been of no use stating them, because the primary interest had not been awakened. Unless and until the interest of the pupil in his subject was stirred, it was of no use giving him the most profound and far-reaching principles. He must be beguiled into such a position.

Professor Baker had been dealing with the proof of the concurrence of the three medians of a triangle, and he (the speaker) had been trying to remember when last he attempted to prove this; he certainly could not remember having gone through any geometrical proof of it since he went to Cambridge as an undergraduate fifty years ago. He had attempted to make a proof in the head while Professor Baker was speaking. This proof that he got, of course, involved the use of parallels in so far as the propositions in the latter half of the first book of Euclid involved the use of parallels; but it involved another proposition which, he would have thought, a schoolboy might appreciate, but which he had never seen enunciated, though it had often been used. If there were three sides of a triangle and the triangle had no area, the third side was the same as the other two. One had flattened it out, it had no area. The proof that he had evolved came to that proposition. He would not say that every schoolboy could appreciate it, but if he gave it to a class he would hope they might appreciate it. Whatever his hopes might have been, however, in teaching, they usually went far beyond his expectations, and his realisations eventually were lower than his expectations. He learned to expect nothing because in the majority of cases he got nothing. The difficulty was to get at something where the attention would be drawn, not because of exernal compulsion for some ulterior purpose, but because of some interest that took the student outside the range of his examinations and led him to care for a thing because of the stimulus to his own knowledge and the creative effort.

GLEANINGS FAR AND NEAR.

428. Who hears now of Abauzit? Yet Newton appealed to Abauzit to decide between him and Leibniz.

429. Burke fond of preserved ginger, and drank great tumblers of cold water. Surely such facts are PRECIOUS!—Sedgwick.

430. For other great mathematicians or philosophers [Gauss] used the epithets magnus, clarus, or clarissimus; for Newton alone he kept the prefix summus.—Ball, *Hist. of Mathematics*, 1901, p. 362.

431. Kepler's suggestion of gravitation with the inverse distance, and Bouillaud's proposed substitutions of the inverse square of the distance, are things which Newton knew better than his modern readers. I have discovered two anagrams on [Newton's] name, which are quite conclusive: the notion of gravitation was *not new*; but Newton *went on*.—De Morgan, *Budget of Paradoxes*, 1872, p. 82.

432. It is the glory of geometry that from so few principles, fetched from without, it is able to accomplish so much.—*Principia*, Preface.

THE NATURE OF AN ALGEBRAIC CURVE AT INFINITY AND THE CONIC OF CLOSEST CONTACT AT INFINITY.

By E. H. Smart, M.A.

The recent discussion in the *Gazette* on the subject of asymptotes to algebraic curves has tempted me to offer a short account of a method of treatment which I have found useful in my teaching, in forming clear conceptions of the nature, not only of the contact of a curve with its asymptotes, but also of such singularities at infinity as the curve possesses. Incidentally the method affords an expeditious means of determining conics of closest contact at infinity—a subject upon which most text-books are discreetly silent—including parabolic asymptotes as a particular case.

It seems to me that the whole question cannot be properly approached without the introduction of homogeneous coordinates, but a very slight acquaintance with these is all that is required. For the rest, as regards technique, I make use of successive approximations at a point in finding the form of the curve at the point, as this seems to me a straightforward method of attack and much to be preferred from the point of view of accuracy to any treatment which involves the use of undetermined coefficients.

1. *The nature of an algebraic curve at infinity.*

Write the equation of the curve as usual in the form

$$u_n + u_{n-1} + u_{n-2} + \ldots + u_1 + u_0 = 0,$$

where u_r is homogeneous of degree r in x and y.

Making this homogeneous by the introduction of z we obtain

$$f(x, y, z) \equiv u_n + u_{n-1}z + u_{n-2}z^2 + \ldots + u_0 z^n = 0; \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$z=0$ is the equation to the line at infinity, and cuts the curve (1) where $u_n=0$, n straight lines cutting $z=0$ in the same number of points at infinity.

We may suppose the curve projected so that xyz becomes a finite triangle. To revert to Cartesians we reverse the process, projecting $z=0$ to infinity and putting $z=1$.

Consider the following cases:

I. *u_n has an unrepeated linear factor $y - \mu x$.*

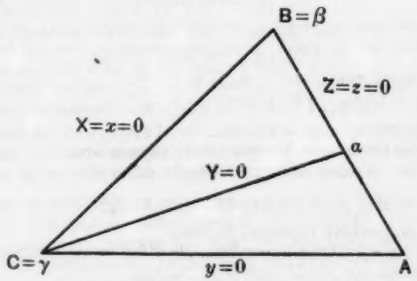

Fig. 1.

Take a new triangle of reference $\alpha\beta\gamma$, defined by the equations

$$x \equiv X, \quad y - \mu x \equiv Y, \quad z \equiv Z.$$

The equation (1) becomes

$$f(X,\ Y+\mu X,\ Z)=0,$$

which may be written at length as

$$YV_{n-1}+U_{n-1}Z+U_{n-2}Z^2+\ldots=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where V_{n-1}, U_{n-1}, U_{n-2}, etc. are homogeneous in X and Y and of the degree indicated by the suffix.

If we put $X=1$ in this, we are supposing the system to revert to Cartesian axes and $X=0$ is now the line at infinity.

We may approximate at the new origin, and for that purpose Y and Z are small.

The first approximation is evidently of the form $Y-aZ=0$, these being the only first degree terms in Y and Z.

The second approximation is of the form $Y-aZ=bZ^2$, showing that this line is a simple tangent to the curve.

II. *u_n has a factor $(y-\mu x)^2$ and u_{n-1} a factor $(y-\mu x)$.*

Transferring as before, the equation of the curve becomes

$$Y^2V_{n-2}+YW_{n-2}Z+U_{n-2}Z^2+U_{n-3}Z^3+\ldots=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Putting $X=1$ as above, the first approximation is of the form

$$a_0Y^2+b_0YZ+c_0Z^2=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

two lines through a touching the projected curve, *i.e.* two parallel asymptotes in the original curve.

The terms of next higher order are

$$a_1Y^3+b_1Y^2Z+c_1YZ^2+d_1Z^3,$$

so for each factor of the quadratic (4) in Y and Z we get a second approximation of the form

$$Y+pZ+qZ^2=0,$$

showing that in this case we get a double point (node, cusp or conjugate point) at infinity.

III. *u_n has a factor $(y-\mu x)^2$ and u_{n-1} does not contain $y-\mu x$ as a factor.*

Transforming as before, the curve is

$$Y^2V_{n-2}+U_{n-1}Z+U_{n-2}Z^2+\ldots=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Putting $X=1$ and approximating at a as before, we have if

$$(V_{n-2})_{x=1}=a_0+a_1Y+\ldots,$$
$$(U_{n-1})_{x=1}=b_0+b_1Y+\ldots,$$
$$(U_{n-2})_{x=1}=c_0+c_1Y+\ldots,$$

for the 1st approximation $\quad b_0Z=0,$

and for the 2nd $\quad b_0Z+a_0Y^2+b_1YZ+c_0Z^2=0, \quad$ *i.e.* $b_0Z=-a_0Y^2,$

showing that the curve touches the line at infinity.

This is of course the case of the parabolic asymptote.

I shall return to its exact determination in particular cases in a moment.

IV. *Generally to deal with singularities at infinity.*

u_n must be examined for repeated factors.

If such a one be $y-\mu x$, take a new triangle of reference as above and transfer as already indicated.

Put $X=1$ and approximate at the new origin. The nature of the singularity appears immediately.

As a numerical illustration consider the curve

$$x^2y^2=4(y+1)$$

from whose equation it is clear that we have to consider what happens at $(\infty, 0)$ and $(0, \infty)$.

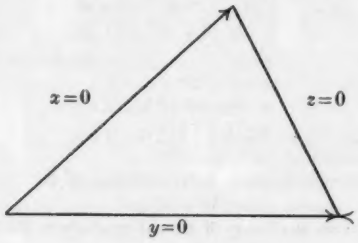

FIG. 2.

Making it homogeneous, we have

$$x^2y^2 = 4yz^3 + 4z^4.$$

For $(\infty, 0)$ put $x=1$, and the approximate form of the curve is

$$y^2 = 4z^4 \quad \text{or} \quad y = \pm 2z^2, \text{ a tacnode.}$$

For $(0, \infty)$ put $y=1$, we have for approximate form

$$x^2 = 4z^3,$$

a *cusp*, the tangent at which is $x=0$.

Similar treatment shows that the curve

$$x^2y^2 = 4(y+4)(y-1)$$

has at $(1, 0, 0)$, *i.e.* $(\infty, 0)$ on the original axes, a tacnode given by $y^2 + 16z^4 = 0$, *i.e.* an isolated point, and at $(0, 1, 0)$, *i.e.* $(0, \infty)$ a node given by

$$x = \pm 2z(1 + \tfrac{3}{2}z)$$

in consonance with the fact that this curve has parallel asymptotes $x = \pm 2$.

For higher singularities it is seen that the method reduces the determination of the compound singularity at any point on the line at infinity to that of the curve $y^q = \mu x^p (p > q)$ at the origin, and as this is known * we have a complete specification of the singularities at infinity on the given curve.

2. *The conic of closest contact at infinity.*

Two examples will be taken to illustrate the method.

Consider the quintic curve $xy^3 = (x+1)^2y^2 + x^5$.

At infinity we have $x^3(x^2+y^2) = 0$. The behaviour at the circular points is purposely omitted and attention will be confined to the point $(0, \infty)$.

Making the equation homogeneous as usual and putting $y=1$, we have a node at this point whose tangents are $x=0$, $z=0$. Regarding this point as origin, we have therefore two conics of closest contact whose tangents at the origin are these lines.

Let the first of these have for its equation

$$x = Ax^2 + 2Hxz + Bz^2.$$

Approximating successively, we have

$$\begin{aligned} x &= Bz^2, \\ x &= Bz^2 + 2HBz^3, \\ x &= Bz^2 + 2Hz(Bz^2 + 2HBz^3) + AB^2z^4 \\ &= Bz^2 + 2HBz^3 + B(4H^2 + AB)z^4. \quad \text{.....(6)} \end{aligned}$$

* It is composed of $\frac{1}{2}(p-3)(q-1)$ double points, $\frac{1}{2}(p-3)(p-q-1)$ double tangents, $(q-1)$ cusps and $(p-q-1)$ inflexions.

The curve when $y=1$ is

$$xz=(x+z)^3+x^5 \quad \text{or} \quad xz(1-3z)=z^3+3x^2z+x^3+x^5,$$

$$x=\frac{z^3}{1-3z}+\frac{3x^2}{1-3z}+\frac{x^3+x^5}{z(1-3z)},$$

whence successively

$$x=z^3,$$
$$x=z^3+3z^5,$$
$$x=z^3+3z^5+12z^4. \quad \text{.....(7).}$$

Comparing (6) and (7), we have $B=1$, $2H=A=3$.
The conic is therefore $x=3x^2+3xz+z^2$.
Making the equation homogeneous by insertion of y,

$$xy=3x^2+3xz+z^2.$$

Finally, putting $z=1$, the conic of closest contact in Cartesian coordinates becomes

$$x(y-3)=1+3x^2.$$

Secondly, consider the conic of closest contact whose tangent is $z=0$. This will be a parabola of course *in its final form.*
Assuming for its equation

$$z=Ax^2+2Hxz+Bz^2$$

(H^2 here is *not* equal to AB), and approximating for z in powers of x as in (6),

$$z=Ax^2+2HAx^3+A(4H^2+AB)x^4. \quad \text{.....(8)}$$

From the curve $xz=(x+z)^3+x^5$,

$$zx(1-3x)=x^3+3xz^2+z^3+x^5,$$

$$z=\frac{x^3}{1-3x}+\frac{3z^2}{1-3x}+\frac{z^3}{x(1-3x)}+\frac{x^4}{1-3x},$$

and successively

$$z=x^3,$$
$$z=x^3+3x^5,$$
$$z=x^3+3x^5+13x^4. \quad \text{.....(9)}$$

By (8) and (9), $A=1$, $2H=3$, $B=4$.
Inserting y for homogeneity and putting $z=1$ we get for the parabola of closest contact

$$y=x^2+3x+4.$$

To take one more example, consider

$$(x^2+y^2)(y-x)^2+18x^3-2y^2+y-3x=0.$$

The form of the equation shows the existence of a parabola of closest contact the direction of whose axis is $y-x=0$.
As in III. after making the equation homogeneous with z, choose a new triangle of reference, putting $X\equiv x$, $Y\equiv y-x$, $Z\equiv z$.

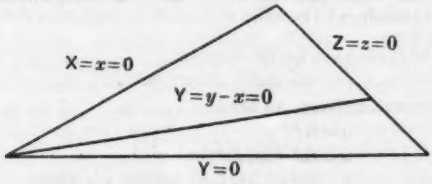

FIG. 3.

We have for the transformed equation

$$(2X^2+2XY+Y^2)Y^2+18X^3Z-Z(Y+X)^2Z^2+(Y-2X)Z^3=0.$$

Put $X=1$ and approximate; we have

$$0=\underset{u_1}{18Z}+\underset{u_2}{(2Y^2-2Z^2)}+\underset{u_3}{(2Y^3-4YZ^2-2Z^3)}+\underset{u_4}{(Y^4-2Y^2Z^2+YZ^3)},$$

and then successively:

1st approximation $Z=0$,

2nd „ $Z=-\frac{1}{9}Y^2$,

3rd „ $Z=-\frac{1}{9}Y^2-\frac{1}{9}Y^3$,

4th „ $Z=-\frac{1}{9}(Y^2-\frac{1}{81}Y^4)-\frac{1}{9}Y^3-\frac{1}{18}Y^4$

$$=-\frac{1}{9}Y^2-\frac{1}{9}Y^3-\frac{19}{1458}Y^4.$$

Comparing with $Z=AY^2+2HYZ+BZ^2$,

which when expanded by successive approximation gives

$$Z=AY^2+2HAY^3+A(4H^2+AB)Y^4$$

to this order, we have on equating coefficients

$$A=-\tfrac{1}{9},\quad 2H=1;$$

$$\therefore \text{ conic is } Z=-\tfrac{1}{9}Y^2+YZ+\tfrac{83}{18}Z^2.$$

Making the equation homogeneous with X and putting $Z=1$ and reverting to x and y, we have

$$x=-\tfrac{1}{9}(y-x)^2+y-x+\tfrac{83}{18} \quad \text{or} \quad 2(y-x)^2+18(2x-y)-83=0,$$

the required parabola.

To sum up: I think there is need for some precision in the use of the term "parabolic asymptote." A straight line is regarded as a true asymptote only when it has the highest possible contact with the curve: *e.g.* in the case of two parallel asymptotes, the term is applied to the two lines which have three-point contact with the curve at infinity, being in fact the tangents to the curve at the infinite node.

To be consistent the term "parabolic asymptote" should be applied only to the parabola of closest contact at infinity. A parabola having a lower order of contact should be referred to as a "parabolic approximation" merely.

My view of the matter also agrees with Mr. Green's contention in the December number of the *Gazette*, that it is incorrect to regard two parallel asymptotes as a particular case of a parabolic asymptote, as such doctrine would be in direct conflict with the accepted theory of Higher Plane Curves.

E. H. Smart.

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433. In a discourse by Sir William Jones, addressed to the Asiatic Society, Feb. 24, 1785, is the following passage: "One of the most sagacious men in this age, who continues, I hope, to improve and adorn it, Samuel Johnson, remarked in my hearing, that if Newton had flourished in ancient Greece, he would have been worshipped as a divinity." *Malone.*—Boswell's *Johnson* (Croker, 1853), footnote, p. 218.

434. Robertson said: "One man has more mind, another more imagination than another." Johnson: "No, sir; it is only, one man has more mind than another. He may direct it differently; he may, by accident, see the success of one kind of study, and take a desire to excel in it. I am persuaded that had Sir Isaac Newton applied to poetry, he would have made a very fine epic poem."—Boswell's *Johnson* (Croker, 1853), p. 274.

435. Robert the Good, titular King of Naples (d. 1480), "was like the algebraist who, after learning some beautiful verses, asked what they proved." —Thomas Campbell's *Life of Petrarch.* [This seems to be the earliest form of the saying attributed somewhere by someone to Newton, about two and a half centuries later.]

ON THE APPROXIMATION TO IRRATIONAL NUMBERS BY RATIONALS.

By A. C. Heath, B.A.

The problem of deciding whether a given number is irrational or not is frequently one of some difficulty, as in the proof that π is irrational. Several numbers which are very probably irrational have never been proved so, *e.g.* γ, e^{π}. However, we have one quite simple test for deciding whether a number is irrational or not when we have the number expressed as a continued fraction, viz. if the continued fraction terminates, the number is rational; if not, irrational. In what follows we assume our irrational number given as a continued fraction:

$$\theta = b_0 + \frac{1}{b_1+}\,\frac{1}{b_2+}\,\frac{1}{b_3+}\cdots$$

or shortly $$[b_0 b_1 b_2 b_3 \ldots].$$

If after some certain b the b's reappear, we have a periodic continued fraction (*cf.* recurring decimal).

In this case we write shortly

$$\theta = [b_0 b_1 \ldots \overline{b_k b_{k+1} \ldots b_m}]$$

meaning $$[b_0 b_1 \ldots b_k b_{k+1} \ldots b_m b_k b_{k+1} \ldots \ldots b_m b_k \ldots].$$

Thus the irrational $\dfrac{1+\sqrt{5}}{2} = 1 + \dfrac{1}{1+}\,\dfrac{1}{1+}\cdots$ is written $[\overline{1}]$.

If $\dfrac{p_n}{q_n}$ denote the n^{th} convergent to θ, and θ_{n+1} the n^{th} complete quotient, then $$\theta = [b_0 b_1 \ldots b_n \theta_{n+1}],$$ and we recall the formula

$$\theta = \frac{p_n \theta_{n+1} + p_{n-1}}{q_n \theta_{n+1} + q_{n-1}}.$$

Clearly we should expect that

$$\left| \theta - \frac{p_n}{q_n} \right| < \frac{1}{q_n},$$

but more than this can be proved; for, using the above,

$$\theta - \frac{p_n}{q_n} = \frac{p_n q_{n-1} - q_n p_{n-1}}{q_n(q_n \theta_{n+1} + q_{n-1})} = \frac{(-1)^n}{q_n^2(\theta_{n+1} + \phi_{n+1})},$$

where $$\phi_{n+1} = \frac{q_{n-1}}{q_n}.$$

Hence our first theorem:

$$\left| \theta - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2},$$

and so by increasing q_n we can approximate to θ as closely as we wish.

The question arises, Is there any c for which

$$\left| \theta - \frac{p}{q} \right| < \frac{1}{cq^2},$$

where for brevity the suffix n is dropped, is true for infinitely many values of n?

This can be looked at in two different ways. One, chiefly studied by Perron and continental writers, is to fix θ and then find the greatest c for which the inequality is true, a number denoted by $M(\theta)$. The other is to fix c and find for what classes of θ's it is true. Looking at it from the second point of view, it is known * that if $c < 3$, the inequality is true for all θ's with a finite number of exceptions (including their equivalent numbers), and that if $c \geqslant 3$, the number of exceptions is infinite.

We first ask, What is the greatest value of c for which there are no exceptions, *i.e.* for which the inequality holds for all θ's?

The answer to this was given by Hurwitz, who shewed that

$$\left| \theta - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}$$

is true for all θ's and infinitely many n's by proving that among any three consecutive convergents to θ at least one has the required property.

Perron has given the following elegant proof on these lines.

We know, as above, that

$$\theta - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n^2(\theta_{n+1} + \phi_{n+1})},$$

where $$\phi_{n+1} = \frac{q_{n-1}}{q_n}.$$

Thus Hurwitz's result is true if for one of any three consecutive values of n,

$$\theta_{n+1} + \phi_{n+1} \geqslant \sqrt{5},$$

i.e. we must prove that if $\theta_{n+1} + \phi_{n+1} \leqslant \sqrt{5}$, ...(1)

and $\theta_{n+2} + \phi_{n+2} \leqslant \sqrt{5}$, ...(2)

it necessarily follows that $\theta_{n+3} + \phi_{n+3} \geqslant \sqrt{5}$. .....(3)

Now $$b_m + \frac{1}{\theta_{m+1}} = \theta_m,$$

so $$\frac{1}{\phi_{m+1}} = \frac{q_m}{q_{m-1}} = \frac{b_m q_{m-1} + q_{m-2}}{q_{m-1}} = b_m + \phi_m;$$

$$\therefore \frac{1}{\theta_{m+1}} + \frac{1}{\phi_{m+1}} = \theta_m + \phi_m.$$

So (1) gives $$\frac{1}{\theta_{n+2}} + \frac{1}{\phi_{n+2}} \leqslant \sqrt{5};$$

$\therefore$ (1) and (2) give

$$(\sqrt{5} - \phi_{n+2})\left(\sqrt{5} - \frac{1}{\phi_{n+2}}\right) \geqslant \theta_{n+2}\frac{1}{\theta_{n+2}}, \text{ i.e. } 1,$$

or $$5 - \sqrt{5}\left(\phi_{n+2} + \frac{1}{\phi_{n+2}}\right) \geqslant 0,$$

and ϕ being rational the equality is impossible;

$$\therefore \phi_{n+2}^2 - \sqrt{5}\,\phi_{n+2} + \tfrac{5}{4} < \tfrac{1}{4},$$

or $$\frac{\sqrt{5}}{2} - \phi_{n+2} < \frac{1}{2},$$

i.e. $$\phi_{n+2} > \frac{\sqrt{5}-1}{2}.$$

* Heawood; *l.c. infra*, ser. 2, vol. 20, p. 233.

If (3) were not correct, we should have similarly

$$\phi_{n+3} > \frac{\sqrt{5}-1}{2},$$

and
$$\therefore\ b_{n+2} = \frac{1}{\phi_{n+3}} - \phi_{n+2} < \frac{2}{\sqrt{5}-1} - \frac{\sqrt{5}-1}{2},\ i.e.\ 1. \quad \ldots\ldots\ldots\ldots\ldots(4)$$

But all b's are positive integers, so this cannot be true; therefore (3) is true, proving the theorem.

Hurwitz was able also to prove that this value of c was the best possible, *i.e.* that for $c > \sqrt{5}$ we could always find one θ such that the inequality is not true.

The following elegant elementary proof is given by Perron.

Choose for θ the fraction representing

$$\frac{1+\sqrt{5}}{2},\ \ i.e.\ [\bar{1}].$$

Then if
$$\left|\frac{1+\sqrt{5}}{2} - \frac{p}{q}\right| < \frac{1}{cq^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

we write
$$\frac{1+\sqrt{5}}{2} = \frac{p}{q} + \frac{\delta}{q^2},\ \ \left(\delta < \frac{1}{c}\right).$$

Squaring, and subtracting the result,

$$1 = \frac{p^2}{q^2} - \frac{p}{q} + \frac{2p\delta}{q^3} + \frac{\delta^2}{q^4} - \frac{\delta}{q^2}$$

or
$$q^2 + pq - p^2 = \frac{2p\delta}{q} + \frac{\delta^2}{q^2} - \delta.$$

Substituting $\dfrac{1+\sqrt{5}}{2} - \dfrac{\delta}{q^2}$ for $\dfrac{p}{q}$,

$$q^2 + pq - p^2 = \sqrt{5}\delta - \frac{\delta^2}{q^2};$$

$$\therefore\ |q^2 + pq - p^2| < \frac{\sqrt{5}}{c}.$$

Now if $c > \sqrt{5}$, the right-hand side of the above is < 1, consequently the left-hand side, being an integer, must be zero.

So
$$q^2 + pq - p^2 = 0;$$

$$\therefore\ (2q+p)^2 = 5p^2$$

or $\sqrt{5}$ = the rational number $\dfrac{2q+p}{p}$.

Thus our hypothesis (5) gives a contradiction and the theorem is proved.

We then ask, If we except this particular θ, can we get a closer approximation? The answer (again due to Hurwitz) is that we can: if $\theta \neq [\bar{1}]$,

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\sqrt{8}q^2}.$$

Proceeding as above, we can replace result (4) by $b_{n+2} < 2$, *i.e.* $b_{n+2} = 1$.

Now unless $\theta = [\bar{1}]$, there are infinitely many b's $\neq 1$, so once more we have a contradiction, and our theorem is proved.

Perron enables us to show that this is the best possible result. He gives the theorem

$$M(\theta)=\sqrt{8} \quad \text{for} \quad \theta=[\bar{2}],$$

so, if $c>\sqrt{8}$, it is not true that

$$\left|[\bar{2}]-\frac{p}{q}\right|<\frac{1}{cq^2} \text{ for infinitely many } n\text{'s}.$$

The question now arises, If $\theta \neq [\bar{1}]$ or $[\bar{2}]$, is there a better approximation? Fujiwara proved that one of any three consecutive convergents satisfies

$$\left|\theta-\frac{p}{q}\right|<\frac{1}{\frac{5}{2}q^2}$$

for all irrationals such that $b_{n+1}\geqslant 2$, and Kurosu proved that if $\theta \neq [\bar{1}]$ or $[\bar{2}]$,

$$\left|\theta-\frac{p}{q}\right|<\frac{1}{\frac{29}{10}q^2} \text{ for infinitely many } n\text{'s}.$$

Later Kurosu replaced $\frac{29}{10}$ by $\frac{84+8\sqrt{3}}{33}$ (2·965),

and showed that the theorem was not true for $\frac{6+2\sqrt{3}}{3}$ (3·154).

In 1924 Shibata published his table, which begins:

(1) $M(\theta)\geqslant\frac{\sqrt{221}}{5}$ except for $\theta=[\bar{1}]$, $[\bar{2}]$.

(2) $M(\theta)\geqslant\frac{\sqrt{1617}}{13}$ except for $\theta=[\bar{1}]$, $[\bar{2}]$, $[\overline{2112}]$.

(3) $M(\theta)\geqslant\frac{\sqrt{7565}}{29}$ except for $\theta=[\bar{1}]$, $[\bar{2}]$, $[\overline{2112}]$, $[\overline{211112}]$.

It is easy to prove all these results best possible, *i.e.* to find a θ, not $[\bar{1}]$ or $[\bar{2}]$, such that

$$\left|\theta-\frac{p}{q}\right|<\frac{1}{cq^2}, \quad c>\frac{\sqrt{221}}{5}$$

is not true for infinitely many n's, for since Perron gives

$$M(\theta)=\frac{\sqrt{221}}{5} \text{ for } \theta=\frac{11+\sqrt{221}}{10},$$

it is not true that

$$\left|\frac{11+\sqrt{221}}{10}-\frac{p}{q}\right|<\frac{1}{cq^2}, \text{ when } c>\frac{\sqrt{221}}{5}.$$

Similarly, $$M(\theta)=\frac{\sqrt{1517}}{13} \text{ for } \theta=\frac{23+\sqrt{1517}}{26},$$

and $$M(\theta)=\frac{\sqrt{7565}}{29} \text{ for } \theta=\frac{53+\sqrt{7565}}{58},$$

serve for (2) and (3).

(We could not expect $[\bar{3}]$ to occur among the exceptional irrationals, since

$$[\bar{3}]=\frac{3+\sqrt{13}}{2}>3, \quad \text{and} \quad \left|\theta-\frac{p}{q}\right|<\frac{1}{cq^2}$$

has infinitely many exceptional classes for $c>3$.)

Kurosu tried to find approximations to θ^2. He proved:

(1) $$\left|\theta^2-\frac{p^2}{q^2}\right|<\frac{\theta}{q^2} \text{ for infinitely many } n\text{'s},$$

the above being true for at least one of any two consecutive convergents;

(2) $$\left|\theta^2-\frac{p^2}{q^2}\right|<\frac{3}{4}\frac{\theta}{q^2}+\epsilon, \quad \epsilon\to 0 \text{ as } n\to\infty,$$

is true for infinitely many n's, if $\theta\neq[\overline{1}]$;

(3) $$\left|\theta^2-\frac{p^2}{q^2}\right|<\frac{12}{17}\frac{\theta}{q^2}+\epsilon, \quad \epsilon\to 0 \text{ as } n\to\infty,$$

is true for infinitely many n's, if $\theta\neq[\overline{1}], [\overline{2}]$.

Proceeding in the same manner as for the θ results, it is easy to show that these are not best possible results, but that the correct theorems are

$$\left|\theta^2-\frac{p^2}{q^2}\right|<\frac{2\theta}{\sqrt{5}q^2} \text{ for all } \theta\text{'s}, \quad <\frac{2\theta}{\sqrt{8}q^2} \text{ for all } \theta\text{'s except } [\overline{1}], \text{ etc.}$$

Further, that these results cannot be improved follows as in the above.

In fact we can, with a little trouble, prove

$$\left|\theta^k-\frac{p^k}{q^k}\right|<\frac{k\theta^{k-1}}{\sqrt{5}q^2} \text{ for all } \theta\text{'s}, \quad <\frac{k\theta^{k-1}}{\sqrt{8}q^2} \text{ for all } \theta\text{'s except } [\overline{1}], \text{ etc.},$$

but this does not tell us anything startlingly new, since θ^k is only another irrational, Θ say. In the θ^2 case we have

$$|q^2\Theta-p^2|<\frac{2\theta}{\sqrt{5}}<1 \quad (\theta<1),$$

or replacing q_n by n, there are infinitely many n's such that $[n^2\Theta]$, *i.e.* the integral part of $n^2\Theta$, is a perfect square.

Note. The subject of this paper was attacked incidentally by A. Markoff in 1879 (*Math. Annalen* 15), and except for a paper by A. Hurwitz in 1897 (*Math. Annalen* 39), was neglected until comparatively recently. Readers interested will find papers by O. Perron (2) in the *Sitzungsberichte d. Heidelberger Acad.* for 1921, by J. H. Grace and P. J. Heawood in the *Proc. of the London Math. Soc.* for 1919 and 1921 respectively, and by M. Fujiwara, K. Kurosu, and K. Shibata in the *Tohoku Math. Journal* for 1916 and later years.

A. C. HEATH.

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436. It is hardly possible to form a sufficient estimate of the immense obligation which the world owes . . . to Halley, without whose great zeal, able management, unwearied perseverance, scientific attainments, and disinterested generosity, the *Principia* might never have been published.—Rigaud, *Essay on the First Publication of the Principia.*

437. In a Latin conversation with the Père Boscovich, at the house of Mrs. Cholmondely, I heard (Johnson) maintain the superiority of Sir Isaac Newton over all foreign philosophers, with a dignity and eloquence that surprised even that learned foreigner.—Boswell's *Johnson* (Croker, 1853), p. 218.

438. Turgot, while still at the Sorbonne, addressed a letter to Buffon in which he objected to the great naturalist's theory of the formation and movements of the planets on the ground that any attempt at fundamental explanations of this kind was a departure from "the simplicity and safe reserve of the philosophy of Newton."—Morley, *Crit. Misc.*, "Turgot."

439. In Haydon's great picture of Christ's entry into Jerusalem are figures of Voltaire sneering, Wordsworth adoring, and Newton meditating.

MATHEMATICAL NOTE.

870. [X. 4, b. α.] *The Graphical Solution of Quadratic Equations.*

In his historical note in the March *Gazette* (p. 318 of this volume) Mr. Milne alludes to my "discovery" of the general graphic method of solving quadratics which I advocate. It is an old method, which seems to have been forgotten or overlooked, and I am merely trying to bring it into the forefront. The application to coaxial ranges *is* due to me. The method is useful also in connection with geometrical optics. A. LODGE.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette.*

Sir,

Can any readers of the *Gazette* throw any light on the history or origin of the following rule for reckoning products between 6×6 and 10×10, inclusive, on the fingers? Canon J. M. Wilson says he has long known it, but does not know how it came to his knowledge or where it originated.

He says the rule is known as *Regula Stultorum.* It seems to have been devised for, and used by, mediaeval monks who could not learn the multiplication table above 5 times.

They numbered the fingers of each hand from 6 to 10, starting with the thumb. Then, supposing they wanted to multiply 8 by 9, they touched the 8th finger of one hand to the 9th finger of the other, then counted and *added* all the fingers above and including the touching ones, for the first or ten digit; and *multiplied* together those beyond, to give the second or unit digit. Thus in this case, there are 7 fingers to be added and 2×1 to be multiplied: result 72.

[*N.B.*—6×6 and 6×7 require 'carrying,' the first being $20+4\times4$, and the second $30+4\times3$; but these are the only two cases.]

It must have been a fascinating discovery, and though it is called the dunce's rule, it was certainly no dunce who invented it, unless it were Duns Scotus himself.

How few people know the multiplication table up to 20×20! While we are on 'aids to multiplication,' the following rule, when both factors are in the 'teens,' make all such products easy to obtain mentally:

(one factor + the unit of the other) × 10 + product of the units
= the product required.

Thus: 19×17 [$=260+63$] $=323$, 19×19 [$=280+81$] $=361$, etc., the middle work being mental.

We might hear at our next annual meeting how it 'catches on' with school boys and girls. A. LODGE.

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440. Jenner writes of a Sir Isaac Newton, Regius Professor of Physic at Cambridge:—"he has out block-headed all his predecessors."

441. (Anne) had already gladded Oxford with her presence, and in 1705 she conceded to Cambridge the costly honour of a royal visitation. A royal visit to a University is, or might be called, a *dunces' holiday*, for then degrees are conferred on all whom royalty appoints, without the statutable qualifications and exercises. Upon this occasion Newton knelt down, plain mister, and arose Sir Isaac. It is the glory of knighthood that such a man deigned to accept it, but it must have been a whimsical spectacle to see a woman holding a sword in an assembly of parsons, to bestow upon a man of peace, an order essentially military.—Hartley Coleridge, *Northerners.*

REVIEWS.

(I.) The Methods of Modern Navigation. By EDWARD J. WILLIS. Pp. 69. $1.00. 1925. (New York: D. Van Nostrand.)

(II.) Spherical Analytical Geometry, being an Appendix to **The Methods of Modern Navigation.** By EDWARD J. WILLIS. Pp. 17. n.p. 1926. (R. Massie Nolting, Richmond, Va., U.S.A.)

All modern methods of getting position at sea are modifications of the one commonly known as Sumner's Method, which was published in 1847 by T. H. Sumner, a Master in the American Mercantile Marine. The method had, however, been used in the Royal Navy previous to this date, and was published by Capt. R. Owen, R.N., some years before Sumner's pamphlet appeared.

The Position Line Method which is now universally adopted was a valuable improvement on the original proposed by Marc St. Hilaire. In this method the calculated Altitude of a heavenly body from an assumed Dead Reckoning position is corrected by the Altitude actually observed. The difference between the two Altitudes (calculated and observed) enables the Navigator to plot on the chart a Line of Position, which is really a short arc of Sumner's Circle, if he can ascertain the True Bearing of the heavenly body. Previous to the publication of Azimuth Tables in 1866, the work of calculation was almost doubled, but since that date the Position Line can be readily drawn without further calculation. A second observation of Altitude of another heavenly body or of the same body after an interval of a few hours enables the exact position of the ship to be ascertained by the intersection of two Position Lines. If the ship has moved during the interval between the two sights, the "run" between the observations must be plotted, and, as the author of *The Methods of Modern Navigation* vividly puts it, "It is as if the ship took this line (the first Position Line) like a bone in her mouth, carried it forward to the place where the next sight was taken and then deposited it on the ocean."

The author of this book is not a professional seaman but an engineer, and he has approached the subject of Navigation in a way that ought to appeal to any one who is not slavishly attached to one stereotyped method. The fundamental formula of Navigation is the equation connecting one side of a spherical triangle with the other two sides and the included angle; this equation he establishes analytically in a very simple way by using the direction cosines of the lines from the centre of the Earth to the ship's position and the geographical position of the heavenly body. Throughout the book Spherical Trigonometry is kept in the background. This and the application of Differential Calculus are distinctly novel features.

The Position Line Method depends on the D.R. position being somewhere in the neighbourhood of the true position, and in consequence the differences between the D.R. Latitude, Longitude and the *calculated* Altitude and the true Latitude, Longitude and *observed* Altitude are all so small that they can be treated as differentials. If we denote the D.R. Latitude, Longitude and calculated Altitude by L, t and A, and the True Bearing of the observed heavenly body by Z, we have:

$$\delta A = \sin Z \,.\, \delta t \cos L + \cos L\, \delta L,$$

when δA, δL and δt are the corrections to be applied to obtain the observed Altitude and the correct Latitude and Longitude. The author regards this as the equation of the Position Line with reference to axes drawn E.-W. and N.-S. through the D.R. position, the coordinates of points on the line being differences of Departure and Latitude. It is, of course, the familiar equation of a straight line in terms of the perpendicular from the origin, and the angle this perpendicular makes with the y-axis.

The analytical method which is employed in dealing with this equation and finding its point of intersection with the second Position Line is of interest to the mathematician; though all the quantities involved may be calculated with sufficient accuracy by slide rule, yet, as the author points out, great care must be taken to give the correct signs to the trigonometrical functions. But to the practical navigator simplicity of method and reliability are of prime

importance, and all difficulties vanish when it is simply a matter of plotting the Position Line on the chart; this can be done immediately Z and δA are known. Needless to say the accuracy of the plotted position is as a rule as good as the data warrant.

In the Appendix the author elaborates in detail a method hinted at in the main work. Taking Latitude and Longitude as coordinates he gives the equation connecting the Latitude and Longitude of a place with the Longitude of the point where the Great Circle through the place meets the Equator, and the angle it makes with the Equator. This equation he regards as the equation of the Great Circle through the place cutting the Equator at a given angle. He works out three examples and a short account of one of these will indicate his method.

A ship receives radio signals from two Wireless Stations and wishes to find her position. These signals give the bearings of the two Stations from the ship. So far as is known these signals travel along Great Circles. The equation of the Great Circle passing through the first W/T station and cutting the D.R. *Longitude* at the given bearing angle is written down. Then a similar equation is written for the Great Circle passing through the W/T station and cutting the D.R. *Latitude* at the given bearing angle. The line joining the two points obtained is a Position Line for the ship. Repeating the process for the second W/T station a second Position Line is obtained and the intersection of the two gives the position.

The method presupposes that the ship has *her own directional* W/T apparatus. If, as is more frequently the case, the ship has to depend on her bearing *as signalled from* the W/T station, the process of finding the Position Line is greatly simplified, but is not reliable if the distance of the ship from the station is more than 100 nautical miles. The reason for this is that the bearing may be more than 1° in error and this error is greatly increased if the distance of the ship from the station is large.

The printing and diagrams are both excellent. The setting out of the paragraphs, however, leaves much to be desired, so that it is difficult to follow the argument. The author has obviously attempted to compress his work into as small a compass as possible; headings to the paragraphs would be of much assistance and would make a strong appeal to the reader. The author passes from one application to the next without giving any indication of the object he has in view. A little care and precision in setting out his paragraphs would have obviated this fault

R. M. M.

The Geometry of the Complex Domain. By J. L. Coolidge. Pp. 242. 21s. 1924. (Clarendon Press.)

Curiosity is whetted when a slender volume bears a title which might cover half a dozen branches of geometry. Prof. Coolidge deals with three subjects, the representation of complex numbers and points in real space, the geometry of families of complex points and figures, and the development of von Staudt's theory identifying complex elements with elliptic involutions; the first two subjects are interwoven and occupy seven of the eight chapters.

The account of the discovery in geometry of a justification for the imaginary variable that is persistently demanded by algebra includes with the familiar names of Wessel, Argand, and Gauss those of several other writers who attacked the problem or made significant contributions to its solution, and ends with Riemann's device of substituting the sphere for the plane in order to contract infinity into a single point. Of work of only historical interest some mathematical details are given, but there is no attempt to expound the geometrical interpretation that every student now learns as a matter of course.

It might seem that when the problem of representing a complex number is solved, that of representing a complex point is solved also, for in any number of dimensions if $x_1, x_2, \ldots x_n$ are the coordinates of a "point" P, this complex "point" is completely represented by the group of ordinary points $P_1, P_2, \ldots P_n$, which correspond individually to the numbers $x_1, x_2, \ldots x_n$. But to see how little such a representation can convey, we have only to remark that it is equally valid in real space, where $P_1, P_2, \ldots P_n$ belong each to a single line. The study of the point-pairs (P_1, P_2) for which the distances of P_1, P_2 from

fixed points O_1, O_2 in fixed lines in which P_1, P_2 are situated satisfy the relation $O_1P_1^2 + O_2P_2^2 = c^2$, may be analytically equivalent to the geometry of the circle, but it is not psychologically equivalent to the third book of Euclid.

The point in the complex plane depends on four real numbers and cannot be represented by a single point in space of fewer dimensions than four. To be confined to three dimensions, a representation must be either by some pair of points or by some geometrical figure dependent on four parameters. The choice of a figure is in a sense arbitrary: any four-parameter figure will do. For example, we might suggest the conic $ax^2 + by^2 + 2gx + 2fy = 1$ or the sphere $(x-a)^2 + (y-b)^2 + (z-f)^2 = g^2$ to represent the complex point $(a+ib, f+ig)$. But apart from what we may call the heterogeneity, that is, the excessive difference between the parts which the different elements of the complex coordinates are called upon to play, which makes us doubt whether such representations can give aesthetic satisfaction, we must ask in any case what family of figures will represent the family of complex points that compose a complex line, and it is only if the family of figures can be recognised as one which we know or are willing to study on its own account, that the representation is one from which we can expect to derive profit. Even with this limitation there is plenty of room for choice, and the history of the representation of complex points is an account not of successive approaches to a solution in any sense final, but of different suggestions each with its own range of interest. Prof. Coolidge tells us briefly, for example, of the "supplementaries" by which Poncelet gave complex points to a real conic, and of the intersections of minimal lines by which Laguerre replaced the complex points of a real plane, and he shows in many cases, what the authors of such suggestions were not concerned to show, the relation in terms of real geometry between one device and another. He has not been afraid to condense a criticism into a single clear-cut sentence, or to enliven his pages by personal touches, as in his reference to Marie, who "possessed the knack of quarrelling with his contemporaries almost to the point of genius." Any lover of pure mathematics, without previous knowledge of complex space or of the correlations in real space to which he will be introduced, will find the historical parts of Prof. Coolidge's book the pleasantest light reading imaginable.

As has been hinted with reference to the straight line, when a representation of a point has been chosen, the representation of the elementary geometrical figures follows automatically, to be studied in whatever detail we like. But it has long been recognised that the complex domain presents problems which are not the immediate generalisations of any problems in real geometry. For example, two conics S', S'' in the complex plane determine a pencil $\lambda S' + \mu S''$; if such propositions as that one conic of the pencil passes through any given point, and that the pencil includes two parabolas, are to remain verbally true, the ratio $\lambda : \mu$ must be a complex parameter, not a real one, and the pencil is a doubly infinite family. Does the pencil stratify naturally into singly infinite families, and if so, what are the properties of such families? Again, if a complex curve is to be formally an extension of a real curve, it must be the aggregate of points dependent on a single complex variable; this is a doubly infinite aggregate, and so also is the aggregate of points dependent on a pair of real variables, but it is only when subject to special conditions, which are easily found, that an aggregate of the second kind is reducible to the first form. In brief, the geometry of complex points dependent on real parameters is a subject to which the geometry of curves and surfaces, of congruences and complexes, in real space, is so far from serving as an introduction that Segre's initial paper, now classical, rightly bears the title *Un nuovo campo di ricerche geometriche*. This is the second subject of Prof. Coolidge's book.

In this work we soon find that every complex number is accompanied everywhere by its conjugate, and we are inevitably prejudiced against the subject. Surely to the mathematician the beauty of the complex number is that he can operate without dissecting it; when decomposition sets in, vitality has gone, and whether the number is split into its real and imaginary parts explicitly or is resolved implicitly by perpetual reference to its conjugate, its reason for having existed is hard to find. The only answer to this criticism is the adequate one that in fact some mathematicians have found the subject attractive, and

the reader will be grateful to Prof. Coolidge for explaining their outlook even if he does not feel the fascination.

Any theorem relating to a family of complex points will have its equivalent in real space in terms of any representation of the complex point, and for this reason the second subject is not separated from the first. As a result, there is an uncommon alternation in the mathematical level of the argument: between pages introducing Hermitian forms and pages discussing a metric which leads naturally to noneuclidean geometry come others with only the most elementary algebra and trigonometry. Previous knowledge is seldom demanded, and the reader who begins by enjoying the gossip and the trivialities may well be drawn insensibly into appreciating the serious mathematics.

The author's function varies with the nature of the work. When not the historian, he is giving the first systematic account of his subject. If the main lines have been laid down by others, and notably by Segre and Study, he has performed excellently the task of coordinating the material and of filling in the gaps between one paper and another, and in so doing he has made some elegant contributions to the theory.

The first three-quarters of the book do not get beyond the complex plane, and within this range the discussion has a measure of completeness. But in the geometry of complex space, except for the analytical extension of properties of real space, a beginning has hardly been made. A figure representing a point in complex space must involve six independent parameters; it is easy to suggest such a figure, but a theorem concerning a family of complex points needs for its interpretation a theorem concerning a family of figures, and, however important such a theorem may be, it is not likely to be so familiar as to illuminate the complex domain. In dealing with the complex line, Prof. Coolidge can dismiss a whole batch of results as "merely restatements of familiar theorems about orthogonal circles," but when he wishes to discuss the isotropic plane, he turns directly to analytical geometry and cannot suggest that even the simple theorem that every perimeter is zero, should be foreseen from any theorem on real circles or real point-pairs. Dependence on algebra in this connection is inevitable, but there seems to me to be a sad lack of mathematical finish about the pages devoted to the isotropic plane: to avoid the peculiarities of coordinate geometry in the plane by relying on an irrelevant three-dimensional frame is to make no pretence at understanding the subject. The conception of the two aspects of an isotropic plane is not introduced, the divergence is so defined that a negative sign comes unnaturally into the formulae, and the theorem that the locus of a point which subtends a constant divergence at two fixed points is a circle is unaccountably omitted.

In adding a chapter developing von Staudt's theory in detail, Prof. Coolidge has perhaps made a mistake. After the analytical method which climbs steadily from dimension to dimension, this theory in which a point and a plane are defined separately and a line is introduced as the intersection of two planes, has an artificial air, and when we find the theorem that if two points of a line are in a given plane the whole line is in that plane to be not an obvious inference from the definitions, but the culmination of a long and tiresome proof, we cannot help feeling that, in the unreplaced metaphor of a vanishing age, the cart is in front of the horse. At least it is certain that only a few of the readers who will enjoy the earlier chapters will be interested in the actual study of sets of axioms on which geometries can be built, and for these few there are already available not merely other accounts of the theory of throws, but Prof. Coolidge's own statement of the foundations of the theory in his treatise on noneuclidean geometry.

Of the whole volume two complaints must be made. Misprints are unpardonably numerous, and the author is as ready as in his other books to abandon one topic for another without giving a hint, typographical or verbal, that the train of thought is not consecutive. In allowing a reader to be bewildered in one way or the other again and again, Prof. Coolidge does himself a grave injustice, for his well-planned account of a little-known field of geometrical investigation only just fails to be as enjoyable as it is instructive. Lastly, if only for the sake of the historical part of the work an index is badly wanted.

E. H. N.

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ERRATA.

IN Prof. Lodge's article on the graphic solution of quadratic equations (March number):

P. 315, *interchange* Fig. 5 *and* Fig. 6;

P. 316, ll. 4 ↑ and 2 ↑, *for OA read OB;*

l. 3 ↑ *for MR and AR read MC and BC.*

[In reading proofs, I altered the lettering of the diagram (Fig. 7) to that of Fig. 9, and rashly *postponed* attending to the text. Actually the simplest course now is to restore *A*, *R* to the diagram in place of *B*, *C*, and to correct *OB* to *OA* in the second line above the figure. To Prof. Lodge and his puzzled readers I tender sincere apologies.—E. H. N.]

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